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Diffusion of weak magnetic fields by isotropic turbulence

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The diffusion of slowly varying, weak magnetic fields by a statistically isotropic and stationary velocity field in a perfectly conducting fluid is studied by Eulerian analysis. The characteristic wavenumber and variance of the velocity field are k_0 and $3v_0^2$, thus defining the eddy-circulation time $\tau_0 = 1/v_0 k_0$. The velocity field is assumed constant on intervals of duration $2\tau_1$ and statistically independent for distinct intervals. Thus the correlation time is τ_1. The α-effect dynamo mechanism in the quasi-linear approximation is corroborated. Both the quasi-linear and the direct-interaction approximations give identical diffusion of magnetic and passive scalar fields in reflexionally invariant turbulence. This result is found to be exact for $\tau_1/\tau_0 \rightarrow 0$ but is demonstrated to be incorrect in general for finite τ_1/τ_0 because of effects of helicity fluctuations. The nature of the failure of the direct-interaction approximation is exhibited by an exactly soluble model system. Analysis based on a double-averaging device shows that long-range, persistent helicity fluctuations in reflexionally invariant turbulence give an anomalous negative contribution to the magnetic diffusivity which depends on the helicity covariance function. We term this the α^2 effect. The magnitude of the effect depends sensitively on the turbulence statistics. If the characteristic scales of the helicity fluctuations are sufficiently larger than τ_0 and $1/k_0$, the magnetic diffusivity is negative, implying unstable growth, while a passive scalar field diffuses normally. On the other hand, a crude estimate suggests that the α^2 effect is small in normally distributed turbulence.

1. Introduction

The evolution of weak magnetic fields in a conducting turbulent fluid has received sustained investigation since the pioneering work of Parker (1955). Recent studies with complete references are Moffatt (1976) and Roberts & Soward (1975). Opposing views are presented by Parker (1971) and Moffatt (1974). Parker argues that a slowly varying mean magnetic field in reflexionally invariant homogeneous turbulence should experience the same long-time diffusivity as a passive scalar field if there is neither Ohmic dissipation nor molecular diffusion. Moffatt considers it unlikely that a diffusivity coefficient for the magnetic field exists at all in the absence of Ohmic dissipation.

The equation for the magnetic field B_i in a perfectly conducting, incompressible fluid with velocity u_i can be written as

$$\partial B_i/\partial t = \partial(u_i B_j - u_j B_i)/\partial x_j, \tag{1.1}$$

with $$\nabla . \mathbf{B} = \nabla . \mathbf{u} = 0.$$

This is identical in form with the inviscid vorticity equation of an Euler fluid. Both Parker and Moffatt work from Cauchy's solution of (1.1):

$$B_i(\mathbf{X}(\mathbf{a}, t), t) = B_j(\mathbf{a}, 0)\, \partial X_i(\mathbf{a}, t)/\partial a_j. \tag{1.2}$$

Here $\mathbf{X}(\mathbf{a}, t)$ is the position at time t of a fluid element whose initial position is $\mathbf{a}$. The corresponding equations for a passive scalar field ϕ which is convected without molecular diffusivity are

$$\partial\phi/\partial t + \mathbf{u} . \nabla\phi = 0 \tag{1.3}$$

and

$$\phi(\mathbf{X}(\mathbf{a}, t), t) = \phi(\mathbf{a}, 0). \tag{1.4}$$

Parker considers the ensemble average of (1.2) for t so large that typical fluid elements have wandered many correlation lengths (eddy diameters) from their origins. He argues that there is then negligible statistical dependence between the displacement and the strain of a fluid element because the element has suffered many random rotations in the course of its long random walk. Parker develops this argument in considerable detail and is led to conclude that the average of (1.2) may be factored to yield

$$\langle B_i(\mathbf{X}(\mathbf{a}, t), t)\rangle = \langle B_j(\mathbf{a}, 0)\rangle \langle \partial X_i(\mathbf{a}, t)/\partial a_j\rangle. \tag{1.5}$$

In homogeneous isotropic turbulence, simple symmetry arguments show that

$$\langle \partial X_i(\mathbf{a}, t)/\partial a_j\rangle = \delta_{ij}. \tag{1.6}$$

Thus (1.5) yields the precise vector analogue of (1.4), averaged, and implies that the diffusion of both magnetic and scalar fields at long times depends solely, and in exactly the same way, upon the distribution of particle displacement $\mathbf{X}(\mathbf{a}, t) - \mathbf{a}$.

Moffatt derives an exact formal expression for the effective eddy diffusivity acting on the magnetic field. If there is homogeneity, isotropy, incompressibility and reflexional invariance, Moffatt's result can be written in the form†

$$\eta_{\text{mag}}(t) = \eta_{\text{scal}}(t) + \Delta\eta(t), \tag{1.7}$$

$$\eta_{\text{scal}}(t) = \frac{1}{3}\int_0^t \langle v_i(\mathbf{a}, t)\, v_i(\mathbf{a}, s)\rangle\, ds, \tag{1.8}$$

$$\Delta\eta(t) = \frac{1}{6}\int_0^t\int_0^t [\langle v_j(\mathbf{a}, t)\, v_j(\mathbf{a}, r)\, \partial v_m(\mathbf{a}, s)/\partial a_m\rangle - \langle v_j(\mathbf{a}, t)\, v_m(\mathbf{a}, r)\, \partial v_m(\mathbf{a}, s)/\partial a_j\rangle]\, ds\, dr. \tag{1.9}$$

Here $v_i(\mathbf{a}, t) = \partial X_i(\mathbf{a}, t)/\partial t$ is the Lagrangian velocity. The quantity η_{scal} is the ordinary turbulent diffusivity derived by Taylor (1921).

Moffatt argues that there is no obvious reason why the double integral in (1.9) should remain finite as $t \to \infty$, let alone vanish. The basic point here is that

$$\int_0^t \partial v_m(\mathbf{a}, s)/\partial a_j\, ds = \partial(X_m(\mathbf{a}, t) - a_m)/\partial a_j$$

† More compact forms of (1.8) and (1.9) are derived in the appendix.

grows without limit as $t \to \infty$, even though its average, as given by (1.6), vanishes. Therefore factoring the averages to give (1.5), or a similar factoring which would give $\Delta\eta(t) = 0$, is a delicate matter requiring careful justification.

The displacement $\mathbf{X}(\mathbf{a}, t) - \mathbf{a}$ is related in a complicated way to the Eulerian velocity field; it carries information from the whole history of the flow. For this reason, the elegant Lagrangian formulae (1.2) and (1.4) perhaps are not the clearest starting point for determining the degree of similarity between the dynamics of magnetic and scalar fields. The present paper deals instead with analysis in the Eulerian frame. Examples are constructed in which there is complete statistical independence of the Eulerian velocity fields in successive short time intervals.† Then the fluctuating magnetic field existing at the end of a given time interval can make no contribution to the mean magnetic field in later time intervals. This permits some simplicities of analysis which are most naturally formulated in Eulerian rather than Lagrangian terms.

In the Eulerian formulation, the evolution of the mean magnetic field can be reduced to that of the mean Green's tensor

$$G_{in}(\mathbf{x}, t; \mathbf{x}', t') = \langle \hat{G}_{in}(\mathbf{x}, t; \mathbf{x}', t') \rangle \tag{1.10}$$

associated with (1.1). The unaveraged tensor $\hat{G}_{in}$ satisfies

$$\hat{G}_{in}(\mathbf{x}, t; \mathbf{x}', t') = B_i(\mathbf{x}, t), \quad B_i(\mathbf{x}, t') = P_{in}(\nabla)\, \delta(\mathbf{x} - \mathbf{x}'), \tag{1.11}$$

where $B_i(\mathbf{x}, t)$ obeys (1.1),

$$P_{in}(\nabla) = \delta_{in} - \nabla^{-2}\partial^2/\partial x_i\, \partial x_n \tag{1.12}$$

is the solenoidal projection operator, and ∇^{-2} is defined by $\nabla^{-2}\nabla^2 f = f$ for well-behaved functions f. Thus $\hat{G}$ is the magnetic field with a particular initial condition.

The present paper has three principal objectives. The first is to suggest, by means of the Eulerian analysis, that the diffusion of magnetic and of scalar fields in reflexion-invariant isotropic turbulence are indeed distinct phenomena; at the least, the magnetic diffusivity appears to be quantitatively smaller than that of the scalar field, and, as Moffatt suggests, it need not exist at all. The anomalous magnetic diffusivity which we find is due to the effects of local helicity fluctuations in turbulence whose mean helicity is zero. Our second objective is to demonstrate that the magnetic diffusion problem lays bare a fundamental inadequacy of analytical turbulence approximations of the direct-interaction type, although these same approximations give excellent results for diffusion of a passive scalar field. The difference is associated with the fact that scalar diffusion conserves scalar variance, while (1.1) does not conserve magnetic energy. Our third objective is to suggest some procedures for future systematic treatment of the magnetic diffusion problem.

Some shortcomings of perturbation-theory-based turbulence approximations for hydromagnetic turbulence were exposed in an earlier paper (Kraichnan & Nagarajan 1967). It was found that the growth or decay of the total energy of

† A similar device is used by Steenbeck & Krause (1969) and by Kraichnan (1968).

a weak magnetic field acted on by turbulence depended upon a competition between dynamo effects and spectral out-sweeping processes. These phenomena may be well estimated individually by the approximations used, but the sign of their algebraic sum, which is crucial in determining growth or decay, cannot be reliably determined. The present paper shows that the same class of turbulence approximations cannot determine whether the mean magnetic field relaxes by a modified eddy diffusion process or shows anomalous, non-diffusive behaviour. The effects of helicity fluctuations in reflexion-invariant turbulence were neglected completely by Kraichnan & Nagarajan because they first show up in the fourth order of perturbation theory, while the structure of the direct-interaction type of approximations is determined by coefficients obtained in second-order perturbation theory. Inclusion of helicity-fluctuation effects may alter some qualitative conclusions about energy transfer in isotropic hydromagnetic turbulence.

2. Diffusion of a passive scalar field

The concept of eddy diffusivity in the convection of a passive scalar field by turbulence is expressed analytically in a simple way by the direct-interaction approximation. In this section we shall review that approximation, and the simpler quasi-linear approximation, in preparation for applying the approximations to the diffusion of weak magnetic fields.

The mean Green's function for the scalar field is

$$G(\mathbf{x}-\mathbf{x}', t-t') = \langle \phi(\mathbf{x}, t) \rangle, \quad \phi(\mathbf{x}, t') = \delta(\mathbf{x}-\mathbf{x}'), \tag{2.1}$$

where ϕ satisfies (1.3). The direct-interaction approximation for G is a non-local diffusion equation (Roberts 1961) which may be written in the form

$$\partial G(\mathbf{x}, t)/\partial t = \frac{\partial}{\partial x_i} \int_0^t ds \int U_{ij}(\mathbf{y}, s)\, G(\mathbf{y}, s) \frac{\partial}{\partial x_j} G(\mathbf{x}-\mathbf{y}, t-s)\, d^3y, \tag{2.2}$$

where
$$U_{ij}(\mathbf{x}-\mathbf{x}', t-t') = \langle u_i(\mathbf{x}, t)\, u_j(\mathbf{x}', t') \rangle \tag{2.3}$$

and homogeneity is assumed. If $\mathbf{x}$ and t are large compared with correlation scales of the turbulence, $G(\mathbf{x}-\mathbf{y}, t-s)$ may be approximated by $G(\mathbf{x}, t)$ in (2.2), so that the equation reduces to the local form

$$\partial G(\mathbf{x}, t)/\partial t = \eta(\infty)\, \nabla^2 G(\mathbf{x}, t), \tag{2.4}$$

where
$$3\eta(t) = \int_0^t ds \int U_{ii}(\mathbf{y}, s)\, G(\mathbf{y}, s)\, d^3y \tag{2.5}$$

and we now make the additional assumption of isotropy. In a Fourier representation,

$$\eta(t) = \frac{1}{3} \int_0^t ds \int U(k, s)\, g(k, s)\, d^3k, \tag{2.6}$$

where $U(k, t)$ and $g(k, t)$ are the respective transforms of $U_{ii}(\mathbf{x}, t)$ and $G(\mathbf{x}, t)$. The energy spectrum function is

$$E(k) = 2\pi k^2 U(k, 0);$$

it satisfies
$$\int_0^\infty E(k)\,dk = \tfrac{3}{2}v_0^2,$$
where v_0 is the root-mean-square velocity in any direction. The initial condition for g is $g(k, 0) = 1$.

Equation (2.2) gives excellent quantitative agreement with computer simulations of diffusion in a random isotropic velocity field whose spectrum is compactly peaked about a wavenumber k_0 (Kraichnan 1970*a*). This agreement holds for randomly time-varying velocity fields in two and three dimensions and for velocity fields in three dimensions which are random in space but frozen in time.

Equation (2.2) is related to a more primitive approximation obtained by simple iteration of (1.3), followed by a statistical-independence approximation. Assume that either there is no fluctuating scalar field at $t = 0$ or that the fluctuating field at $t = 0$ is statistically independent of the velocity field. Formal integration of (1.3), insertion of the resultant expression for $\phi(\mathbf{x}, t)$ back into the right-hand side of (1.3) and averaging then yields
$$\partial\langle\phi(\mathbf{x},t)\rangle/\partial t = \left\langle \mathbf{u}(\mathbf{x},t)\,.\,\nabla\int_0^t \mathbf{u}(\mathbf{x},s)\,.\,\nabla\phi(\mathbf{x},s)\right\rangle ds. \tag{2.7}$$
There is no initial-value contribution to (2.7) because of our assumption and because $\langle\mathbf{u}(\mathbf{x},t)\rangle = 0$. Now make the approximation of splitting the average on the right-hand side into averages over the velocity field and scalar field individually by replacing $\phi(\mathbf{x}, s)$ with $\langle\phi(\mathbf{x}, s)\rangle$. The result is
$$\partial\langle\phi(\mathbf{x},t)\rangle/\partial t = \frac{\partial}{\partial x_i}\int_0^t U_{ij}(0,t-s)\,\frac{\partial}{\partial x_j}\langle\phi(\mathbf{x},s)\rangle\,ds. \tag{2.8}$$
We shall call (2.8) the quasi-linear approximation. It has been made in many contexts, and under many different names. In the hydromagnetic literature it has commonly been called the first-order-smoothing approximation.

Equation (2.8) is local in space, in contrast to (2.2). If $G(\mathbf{y}, s)$ in (2.2) is approximated by $G(\mathbf{y}, 0) = \delta(\mathbf{y})$, then the two equations become the same, provided the initial condition (2.1) is used with (2.8).

If the velocity field is peaked about wavenumber k_0, an eddy-circulation time is defined by $\tau_0 = 1/v_0 k_0$. A correlation time of the velocity field is defined by
$$\tau_1 = \int_0^\infty U_{ii}(0,t)\,dt/3v_0^2. \tag{2.9}$$
If the velocity field obeys the Navier–Stokes equation at moderate Reynolds numbers, $\tau_1 \sim \tau_0$.

In the limit $\tau_1 \ll \tau_0$ (white-noise velocity field), $g(k, t) \approx 1$ for k which contribute appreciably to the integral over $U_{ij}(0, t)$. In this limit, (2.2), (2.4) and (2.8) are equivalent and asymptotically exact, for $\gg \tau_1$. Also, in the limit,
$$\eta(t) = \frac{1}{3}\int_0^t ds\int U(k,s)\,d^3k, \quad \eta(\infty) = \tau_1 v_0^2. \tag{2.10}$$
Near the limit, (2.4), with $\eta(\infty) = \tau_1 v_0^2$, is in error by $O[(\tau_1/\tau_0)^2\,\nabla^2\phi]$. This is easily seen in the case where $\mathbf{u}(\mathbf{x}, t)$ is constant in time on intervals of length $2\tau_1$ and

statistically independent in successive intervals. The error in passing from the exact equation (2.7) is then readily estimated because only that part of the fluctuating scalar field generated during a given time interval of length $2\tau_1$ can contribute to the right-hand side of (2.7) during that interval.

When $\tau_1 \gtrsim \tau_0$, the factor $G(\mathbf{y}, s)$ in the integrand makes an effective cut-off of the integral (2.5) at $t \sim \tau_0$, giving $\eta(\infty) \sim \tau_0 v_0^2$. This is true in particular for frozen velocity fields with $\tau_1 = \infty$ (Kraichnan 1970*a*). Thus the principal improvement offered by the direct-interaction approximation over the simpler quasi-linear approximation is the replacement of τ_1 in the expression for $\eta(\infty)$ by an effective correlation time τ_* given approximately by

$$\tau_* = \tau_1 \quad (\tau_1 \ll \tau_0), \quad \tau_* \sim \tau_0 \quad (\tau_1 \lesssim \tau_0). \tag{2.11}$$

3. Diffusion of magnetic fields by helical isotropic turbulence

Both the direct-interaction approximation and the quasi-linear approximation can be carried out for magnetic fields obeying (1.1) in close analogy to the analysis for the passive scalar field (Lerche 1973*a*, *b*, *c*). There is some added complexity because the magnetic field is a vector and the direct-interaction equations involve the tensor Green's function. If the velocity field is statistically homogeneous and isotropic, the Green's tensor can be expressed in terms of a single scalar and a single pseudo-scalar. Thus

$$G_{ij}(\mathbf{x}, t; \mathbf{x}', t') = P_{ij}(\nabla)\, G(\mathbf{x}-\mathbf{x}', t-t') + \epsilon_{imj}\, \partial H(\mathbf{x}-\mathbf{x}', t-t')/\partial x_m \tag{3.1}$$

or, in the Fourier representation,

$$g_{ij}(\mathbf{k}, t) = P_{ij}(\mathbf{k})\, g(k, t) + ik_m \epsilon_{imj} h(k, t). \tag{3.2}$$

We assume stationarity also in choosing the notation for (3.1) and (3.2). The pseudo-scalars $H(x, t)$ and $h(k, t)$ vanish if there is reflexion invariance.

We shall work out explicitly only the quasi-linear equations. Assume that the velocity correlation has the form

$$U_{ij}(\mathbf{x}, t) = U_{ij}(\mathbf{x}, 0)\, D(t). \tag{3.3}$$

The quasi-linear analysis proceeds in direct analogy to the passage from (2.7) to (2.8). In the present case, we find

$$\partial\langle B_i(\mathbf{x}, t)\rangle/\partial t = L_{in}(\mathbf{x}, t)\int_0^t D(t-s)\langle B_n(\mathbf{x}, s)\rangle\, ds, \tag{3.4}$$

where

$$L_{in}(\mathbf{x}, t) = \langle \nabla_j u_i \nabla_n u_j\rangle - \langle \nabla_n u_i \nabla_j u_j\rangle - \langle \nabla_j u_j \nabla_n u_i\rangle + \langle \nabla_j u_j \nabla_r u_r \delta_{in}\rangle \tag{3.5}$$

and $\nabla_i = \partial/\partial x_i$. Each ∇ in the operator $L_{in}(\mathbf{x}, t)$ acts on everything to its right, and all the u factors have arguments $(\mathbf{x}, t)$.

Isotropy implies

$$\langle u_i u_j\rangle = \delta_{ij} v_0^2, \quad \langle u_i\, \partial u_j/\partial x_n\rangle = \mu\epsilon_{inj}, \tag{3.6), (3.7}$$

and the helicity density is

$$\langle \mathbf{u}\,.\,\boldsymbol{\omega}\rangle = \langle u_i \epsilon_{inj}\, \partial u_j/\partial x_n\rangle = 6\mu. \tag{3.8}$$

Equations (3.6) and (3.7), together with homogeneity and the solenoidal conditions $\nabla . \mathbf{u} = \nabla . \mathbf{B} = 0$, give a simplification of $L_{in}(\mathbf{x}, t)$ to the form

$$L_{in} = v_0^2 \delta_{in} \nabla^2 - 2\mu\epsilon_{ijn} \nabla_j. \tag{3.9}$$

If $\langle B(\mathbf{x}, t)\rangle$ varies slowly compared with the correlation scales of the velocity field, (3.4) then reduces to

$$\partial\langle \mathbf{B}\rangle/\partial t = \tau_1 v_0^2 \nabla^2 \langle \mathbf{B}\rangle - 2\tau_1 \mu \boldsymbol{\nabla} \times \langle \mathbf{B}\rangle. \tag{3.10}$$

Equation (3.11) resembles the equation (Steenbeck & Krause 1969; Moffatt 1970)

$$\partial\langle \mathbf{B}\rangle/\partial t = \lambda \nabla^2 \langle \mathbf{B}\rangle - \alpha' \boldsymbol{\nabla} \times \langle \mathbf{B}\rangle \tag{3.11}$$

for the low-conductivity regime. Here λ is the Ohmic diffusivity and

$$\alpha' = \tfrac{1}{3}\lambda \int_0^\infty k^{-2} F(k)\, dk, \tag{3.12}$$

where $F(k)$ is the helicity spectrum. The present quasi-linear approximation has been discussed by Steenbeck & Krause (1969), Roberts (1971), Lerche & Parker (1973), Moffatt (1974), Roberts & Soward (1975), and others.

Moffatt states that, in conflict with (3.10), $\langle \mathbf{u} \times \mathbf{b}\rangle$ vanishes in the quasi-linear approximation unless $\lambda > 0$. He starts by noting that the quasi-linear approximation for $\lambda = 0$ is equivalent to assuming

$$\partial \mathbf{b}/\partial t = \nabla \times (\mathbf{u} \times \langle \mathbf{B}\rangle), \tag{3.13}$$

where $\mathbf{b} = \mathbf{B} - \langle \mathbf{B}\rangle$ is the fluctuating magnetic field. The four-dimensional Fourier transform of (3.13) is

$$\omega \tilde{\mathbf{b}} = -(\mathbf{k} . \mathbf{B}_0)\, \tilde{\mathbf{u}}, \tag{3.14}$$

where ω is frequency and $\mathbf{B}_0$ is a uniform mean magnetic field, the simplest case. Moffatt concludes from (3.14) that $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{u}}$ are 'exactly in phase', so that the mean electromagnetic force $\langle \mathbf{u} \times \mathbf{b}\rangle$ vanishes. This argument is incorrect because it fails to take into account the initial condition $\mathbf{b}(\mathbf{x}, 0) = 0$ (Roberts & Soward 1975). In fact, if $\mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0$ it follows directly from (1.1), without any approximation, that

$$[d\langle \mathbf{u} \times \mathbf{b}\rangle/dt]_{t=0} = -2\mu \mathbf{B}_0. \tag{3.15}$$

As in the scalar case, (3.10) is asymptotically exact, with error of order τ_1^2/τ_0^2, as $\tau_1/\tau_0 \to 0$. The direct-interaction equations for the magnetic Green's tensor have not yet been solved in detail for isotropic turbulence with non-vanishing helicity. Their form strongly suggests that, again as in the scalar case, they yield behaviour for slowly varying mean magnetic fields which is qualitatively the same as (3.10) if τ_1 in the latter is replaced by the τ_* defined in (2.11). With this replacement, (3.10) asserts, in essence, that the behaviour of $\langle \mathbf{B}\rangle$ is well characterized by the initial behaviour of $\langle \partial^2 \mathbf{B}/\partial t^2\rangle$ at each of a succession of time intervals equal to an effective decorrelation time of the diffusion process. This decorrelation time is the velocity decorrelation time τ_1 if τ_1 is shorter than the eddy-circulation time τ_0 and is τ_0 otherwise. The effective time does not exceed τ_0 even in frozen turbulence because in a time τ_0 a fluid element wanders from one eddy to a statistically independent eddy, an effect which is incorporated in the direct-interaction equations by the presence of the Green's function in the effective diffusivity [cf. (2.5)].

Let us write the altered (3.10) as

$$\partial \mathbf{B}/\partial t = \eta \nabla^2 \mathbf{B} - \alpha \nabla \times \mathbf{B}, \tag{3.16}$$

where $\eta = \tau_* v_0^2$, $\alpha = 2\tau_* \mu$ and we have dropped the angular brackets for notational convenience. The response tensor (3.2) associated with (3.16) is

$$g_{ij}(\mathbf{k}, t) = \exp(-\eta k^2 t)\,[P_{ij}(\mathbf{k}) \cosh(\alpha k t) - i\epsilon_{imj} k_m k^{-1} \sinh(\alpha k t)]. \tag{3.17}$$

The implications of (3.17) are easiest to display if the initial magnetic field is a plane sheet of the form

$$B_1(\mathbf{x}, 0) = B_2(\mathbf{x}, 0) = 0, \quad B_3(\mathbf{x}, 0) = \exp(-x_1^2/4b^2), \tag{3.18}$$

where b is a width parameter. Fourier transformation, the use of (3.17) and transformation back to $\mathbf{x}$ space yields (cf. Moffatt 1970)

$$B_1(\mathbf{x}, t) = 0, \quad B_2(\mathbf{x}, t) = -f(x_1, t) \sin(\alpha x_1/2\eta), \quad B_3(\mathbf{x}, t) = f(x_1, t) \cos(\alpha x_1/2\eta), \tag{3.19}$$

where
$$f(x_1, t) = (4\pi\eta t)^{-\frac{1}{2}} \exp[(\alpha^2 t/4\eta) - x_1^2/4(\eta t + b^2)]. \tag{3.20}$$

Thus the α term in (3.16) gives two phenomena superimposed on the diffusion due to the η term. The first is an overall amplification by the factor $\exp(\alpha^2 t/4\eta)$ and the second is a helical wave structure in the x_1 direction such that the $\mathbf{B}$ vector rotates parallel to the x_2, x_3 plane with wavenumber $\alpha/2\eta$.

If the turbulence approaches maximal helicity (Kraichnan 1973), $\mu \sim v_0^2 k_0$, so that

$$\alpha \sim v_0^2 k_0 \tau_* = v_0(\tau_*/\tau_0).$$

Then
$$\alpha/\eta \sim k_0, \quad \alpha^2/\eta \sim \tau_*/\tau_0^2. \tag{3.21}$$

Thus the wavenumber of the helical wave in this case is $\sim k_0$, independent of τ_*, while the amplification-factor growth rate is proportional to τ_*. For $\tau_* \sim \tau_0$, the growth rate is $\sim 1/\tau_0$.

4. Helicity fluctuations in reflexion-invariant turbulence

If the isotropic homogeneous turbulence has reflexion invariance, then both the quasi-linear approximation and the direct-interaction approximation predict that the diffusion of a weak magnetic field is identical to that of a passive scalar field. The symmetries of this case require that the pseudo-scalar H in (3.1) vanish. The equations for the scalar G in the two approximations are then precisely those given for the scalar G in §2. We wish now to demonstrate that the effects of local helicity fluctuations make this prediction internally inconsistent and may cause profound differences in behaviour between scalar and magnetic fields.

We start by constructing an ensemble of velocity fields such that the isotropic relation (3.6) holds everywhere but (3.7) is replaced by

$$\langle u_i\, \partial u_j/\partial x_n \rangle = \epsilon_{inj}\, \mu(\mathbf{x}, t), \tag{4.1}$$

where $\mu(\mathbf{x}, t)$ is a nearly arbitrary function of $\mathbf{x}$ and t. These properties can be realized by a superposition of randomly placed and oriented flow elements, each element consisting of a pair of linked vortex rings. The rings all have identical size and shape. The axial vortex line of each ring forms a circle; for the two rings of an element, these circles are perpendicular, and each passes through the centre of the other.†

Such flow elements have an intrinsic helicity whose sign is determined by whether the vortex vectors of the two rings go clockwise or counterclockwise around each other. The sign is independent of the orientation of the element. It reverses if the vorticity of one ring is reversed but is unchanged under reversal of both rings (change of sign of the velocity field of the element).

Equation (3.6) is assured if the placement of elements is statistically homogeneous and isotropic, disregarding helicity sign. Equation (4.1) is realized by weighting the relative probabilities of the two possible signs of helicity of an element centred at $\mathbf{x}$ according to $\mu(\mathbf{x}, t)$. Equal probabilities everywhere give $\mu(\mathbf{x}, t) = 0$ everywhere. The verification of (3.6) involves two steps. There is no velocity correlation between distinct elements because the sign of the velocity of each element is random. (Note that reversal of the sign of the velocity of an element corresponds to rotation of the element through 180° about the axis connecting the centres of the two vortex rings.) The diagonal elements of (3.6) then follow immediately from overall homogeneity and isotropy and are obviously independent of the helicity distribution. The vanishing of the off-diagonal elements of $\langle u_i u_j \rangle$ requires that the average of $u_i u_j$ over the orientation of a single element vanishes everywhere in the field of that element unless $i = j$. This can be verified by considering the effects of rotations of the element about and perpendicular to the axis of centres.

If the thickness of the vortex rings is comparable to their diameter and the latter is $\sim 1/k_0$, then the velocity field will be peaked about a wavenumber k_0. A velocity correlation time τ_1 with $\mu(\mathbf{x}, t)$ constant in time can be realized by changing to new, independent realizations at time intervals of length $2\tau_1$. There are restrictions on the degree of variation of $\mu(\mathbf{x}, t)$ with $\mathbf{x}$ and t. Values of $|\mu(\mathbf{x}, t)|$ of order $v_0^2 k_0$ can be realized but only if the scale of spatial variation of $\mu(\mathbf{x}, t)$ is $\gtrsim 1/k_0$. The scale of variation of $\mu(\mathbf{x}, t)$ in time must of course be $\gtrsim \tau_1$ if the ensemble is realized as stated.

If the analysis leading to (3.10) is now repeated, the result is

$$\partial \mathbf{B}/\partial t = \eta \nabla^2 \mathbf{B} - \nabla \times [\alpha(\mathbf{x}, t)\, \mathbf{B}], \tag{4.2}$$

where

$$\eta = \tau_1 v_0^2, \quad \alpha(\mathbf{x}, t) = 2\tau_1 \mu(\mathbf{x}, t),$$

and $\mathbf{B}$ denotes an average over the present ensemble. Our essential step is now to consider a superensemble, or ensemble of ensembles, over which $\alpha(\mathbf{x}, t)$ is a stationary, homogeneous, isotropic random function of $\mathbf{x}$ and t, with zero mean. Then (4.2) is a stochastic equation which may be treated by the quasi-linear and

† See Moffatt (1969) for the relation between helicity and linkage of vortex lines and for other examples of flow elements with intrinsic helicity.

direct-interaction approximations. We take

$$\langle \alpha(\mathbf{x},t)\,\alpha(\mathbf{x}',t')\rangle = A(\mathbf{x}-\mathbf{x}')\,D_2(t-t'), \quad A \equiv A(0), \quad \tau_2 = \int_0^\infty D(t)\,dt, \tag{4.3}$$

where angular brackets now denote averaging over the superensemble.

The quasi-linear approximation to (4.2) is slightly complicated by the damping term $\eta\nabla^2\mathbf{B}$. If the Green's function associated with this term is used to integrate (4.2) formally, then the analogue to (2.7) so obtained is

$$\partial\langle \mathbf{B}(\mathbf{x},t)\rangle/\partial t = \eta\nabla^2\langle\mathbf{B}\rangle + \nabla_x \times \int\left\langle \alpha(\mathbf{x},t)\int_0^t G_0(\mathbf{x}-\mathbf{y},t-s)\,\nabla_y \times [\alpha(\mathbf{y},s)\,\mathbf{B}(\mathbf{y},s)]\,ds\,d^3y\right\rangle. \tag{4.4}$$

where
$$G_0(\mathbf{x},t) = (4\pi\eta t)^{-\frac{3}{2}}\exp(-x^2/4\eta t). \tag{4.5}$$

We now assume that the spectrum of $\alpha(\mathbf{x},t)$ is concentrated about a wave-number k_2, and we restrict τ_2 by

$$\tau_2 \geqslant \tau_1, \quad \tau_2 \ll (A^{\frac{1}{2}}k_2)^{-1}, \quad \tau_2 \ll (\eta k_2^2)^{-1}. \tag{4.6}$$

The first of these three conditions is a consistency requirement which follows from the way we have constructed the superensemble. The second is the condition for validity of the quasi-linear approximation (note that α has the dimensions of velocity and plays a role analogous to that of $\mathbf{u}$ in (2.7)). The third condition ensures that the smearing effects of G_0 in (4.4) can be neglected during the interval of order τ_2 in which correlations in (4.4) are non-negligible.

With (4.6) satisfied, we can replace G_0 by $\delta(\mathbf{x}-\mathbf{y})$ in (4.4), split the average, replace $\langle\mathbf{B}(\mathbf{y},s)\rangle$ by $\langle\mathbf{B}(\mathbf{x},t)\rangle$ and obtain, for $t \gg \tau_2$,

$$\partial\langle\mathbf{B}\rangle/\partial t = \eta\nabla^2\langle\mathbf{B}\rangle + \tau_2\nabla\times\langle\alpha\nabla\times\alpha\rangle\langle\mathbf{B}\rangle. \tag{4.7}$$

Now
$$\langle\alpha\nabla\times\alpha\rangle = A\nabla\times + \langle\alpha\nabla\alpha\rangle\times$$

and the second term vanishes because of homogeneity. Hence, noting that

$$\nabla\times\nabla\times = \nabla\nabla. - \nabla^2, \quad \nabla.\mathbf{B} = 0,$$

we have, finally,
$$\partial\langle\mathbf{B}\rangle/\partial t = (\eta-\tau_2 A)\,\nabla^2\langle\mathbf{B}\rangle. \tag{4.8}$$

Equation (4.8) shows that sufficiently long-range and persistent zero-mean helicity fluctuations in a reflexionally invariant, statistically isotropic velocity field make a negative contribution to the effective eddy diffusivity acting on the magnetic field. If $A \sim v_0^4 k_0^2 \tau_1^2$ (maximal helicity in typical realizations of the subensemble), then

$$\tau_2 A/\eta \sim \tau_2\tau_1 v_0^2 k_0^2 = \tau_2\tau_1/\tau_0^2,$$

which goes to zero as $\tau_2/\tau_0 \to 0$, $\tau_1/\tau_0 \to 0$. This is consistent with the conclusion in §3 that the quasi-linear approximation (3.10) is exact in the limit. (Note that, in (3.10), μ measures the mean helicity, which is zero for the present ensemble.) But if (4.8) is extrapolated to $\tau_1 \sim \tau_0$, as was done with (3.10), then the negative term in the diffusivity is, at the least, comparable to the positive term, since we take $\tau_2 > \tau_1$. If τ_2 is sufficiently larger than τ_1, and k_2/k_0 is small enough that (4.6) holds, then $\eta - \tau_2 A_2$ is negative. This regime can be reached by sufficiently large τ_2

and small k_2 even if A has a value corresponding to much weaker-than-maximal helicity fluctuations. We still assume, in this regime, that the typical wave-numbers of $\langle \mathbf{B} \rangle$ are small compared with k_2. Thus we are led to the conclusion that there exist reflexionally invariant, statistically isotropic velocity fields in which a mean magnetic field experiences a negative net diffusivity, and so grows unstably, while a passive scalar field feels a normal, positive eddy diffusivity.

This conclusion clearly is inconsistent with the prediction of identical diffusivity for scalar and magnetic fields which comes from directly applying the quasi-linear or direct-interaction approximation to the zero-helicity super-ensemble. If the same double-averaging analysis is applied to scalar diffusion, no inconsistency with direct use of the quasi-linear or direct-interaction approximation arises. This is because the eddy diffusivity (2.5) depends only on the diagonal elements U_{ii} and is independent of whether α is zero everywhere, fluctuates or is a non-zero constant.

In the magnetic case, we believe that the correct physics are contained in the double-ensemble analysis; that the derivation of (4.8) is valid if (4.6) is satisfied; and that the suggested difference between the behaviour of scalar and magnetic fields is real. The implication is then that there is a qualitative deficiency in the quasi-linear and direct-interaction approximations when they are applied directly to calculate the magnetic Green's tensor in reflexion-invariant isotropic turbulence. In §5, we shall display the nature of the trouble, and give some support to the present two-stage analysis, by applying the approximations to a simple model problem for which exact solutions are available.

In the quasi-linear approximation, η is a second-order moment of the velocity field while A is a fourth-order moment. Thus the relative magnitude of the negative contribution to the net diffusivity in (4.8) can depend sensitively on the statistics of the turbulence. The superensembles with long-range helicity fluctuations treated in the present section are necessarily non-normal. We wish now to attempt a crude estimate of the effect of helicity fluctuations on magnetic diffusivity in reflexion-invariant, normally distributed turbulence.

The standard rule for reducing fourth-order moments of a normal zero-mean field to a sum of products of second-order moments gives

$$\langle (\mathbf{u} \cdot \boldsymbol{\omega})^2 \rangle = 39\mu^2 - v_0^2 \langle \mathbf{u} \cdot \nabla^2 \mathbf{u} \rangle, \tag{4.9}$$

where the second-order moments have been evaluated from the isotropic relations (3.6)–(3.8). To give an explicit illustration, suppose that the velocity spectrum is confined to a thin shell of radius k_0 in k space. Then $\langle \mathbf{u} \cdot \nabla^2 \mathbf{u} \rangle = -3v_0^2 k_0^2$ and the condition that the helicity should fall within maximal values is

$$|\langle \mathbf{u} \cdot \boldsymbol{\omega} \rangle| = |6\mu| \leqslant 3k_0^2 v_0^2. \tag{4.10}$$

Thus (4.9) yields
$$4v_0^4 k_0^2 \leqslant \langle (\mathbf{u} \cdot \boldsymbol{\omega})^2 \rangle \leqslant \tfrac{51}{4} v_0^4 k_0^2. \tag{4.11}$$

The minimum value in (4.11) corresponds to multivariate-normal, reflexion-invariant velocity statistics. The maximum value can be realized in a non-normal, reflexionally invariant superensemble by combining, with equal weights, normal ensembles with maximal positive and maximal negative helicities. The mean-square helicity fluctuation in the normal, reflexionally invariant ensemble is then $\frac{4}{17}$ of the mean-square helicity fluctuation in the non-normal ensemble. The latter has the same values of v_0^2 and $\langle u^4\rangle/\langle u^2\rangle^2$ as the former but has maximal helicity in each realization.

Now, to estimate the behaviour of the normal, non-helical ensemble with spectrum concentrated at k_0, we take $\tau_2 = \tau_1$ and carry through the super-ensemble analysis with a value of A corresponding to the minimum value in (4.11). Then in (4.8) we find

$$A = \tfrac{1}{3}v_0^4 k_0^2 \tau_1^2, \quad \tau_2 A/\eta = \tfrac{1}{3}(\tau_1/\tau_0)^2. \tag{4.12}$$

The relative magnitude of the negative contribution to the net diffusivity then increases like τ_1^2 and should be maximum for a frozen velocity field. In the latter case we replace τ_1 by $\tau_* = \tau_0$, as in §2. [For a frozen field with concentrated spectrum, the steady-state passive scalar η is given quite accurately by $1{\cdot}0 v_0^2\tau_0$ according to numerical experiments (Kraichnan 1970*a*).] Equation (4.12) suggests that the difference between magnetic and scalar diffusivities in normal, non-helical turbulence should be small if τ_1 is less than, say, $\frac{1}{2}\tau_0$ and that the magnetic diffusivity should be positive and not very different from the scalar value even in the limit of frozen fields. Moreover, we can argue that the present estimates of the negative contribution to diffusivity are overestimates. This is because the third inequality in (4.6) is violated when $\tau_2 = \tau_0$, $k_2 = k_0$ and the smearing effect of G_0 in (4.4) no longer is negligible. However, the crudity, lack of rigour and consequent unreliability of our estimation procedure for normal, non-helical turbulence should be emphasized.

The analysis leading to (4.8) has been carried out in the Eulerian framework. The result is corroborated by Lagrangian analysis starting from Moffatt's (1974) expressions for magnetic diffusivity. The Lagrangian treatment is outlined in the appendix.

5. A model problem

Equation (1.3) for scalar diffusion is a linear equation with the velocity field as a stochastic coefficient function. It conserves the integral of ϕ^2 over all space. The simplest model of such a system is the random oscillator

$$dy_1/dt = ay_2, \quad dy_2/dt = -ay_1, \tag{5.1}$$

where a is a random variable with zero mean. It conserves $y_1^2+y_2^2$. Assume that a is Gaussian and define τ_0 by $\langle a^2\rangle = 1/\tau_0^2$. Let a be piecewise constant on intervals of duration $2\tau_1$ and statistically independent for distinct intervals, with the first interval starting at $t = 0$.†

The off-diagonal elements G_{12} and G_{21} of the average response matrix vanish by symmetry. If $\tau_1 = \infty$, the diagonal elements are exactly (Kraichnan 1961)

$$G(t) = G_{11}(t) = G_{22}(t) = \exp\left(-\tfrac{1}{2}t^2/\tau_0^2\right). \tag{5.2}$$

For finite τ_1, only the average value of y at the end of an interval $2\tau_1$ can contribute to the evolution of the average in the next interval because of the independence

† Strict stationarity can be restored by randomizing the interval end points, thereby introducing complications, but inessential changes, in the results.

condition on a. Hence, from the definition of a response matrix, and the fact that $G_{nm}(t)$ is diagonal, the value of the response function $G(t)$ at the end of the nth interval is

$$G(t) = [\exp(-2\tau_1^2/\tau_0^2)]^n = \exp(-t\tau_1/\tau_0^2) \quad (t = 2n\tau_1). \tag{5.3}$$

The direct-interaction approximation for $G(t)$ in the case $\tau_1 = \infty$ is (Kraichnan 1961)

$$G(t) = (t/\tau_0)^{-1} J_1(2t/\tau_0), \tag{5.4}$$

where J_1 is the Bessel function of order 1. If τ_1 is finite, the same argument applies as for the exact solution, and the result is

$$G(t) = [J_1(4\tau_1/\tau_0)/(2\tau_1/\tau_0)]^n \quad (t = 2n\tau_1). \tag{5.5}$$

Thus both exact and approximate $G(t)$ show essentially exponential decay with t for finite τ_1. They agree asymptotically in the limit $\tau_1/\tau_0 \to 0$, and, for $\tau_1 \sim \tau_0$, both give an exponential decay rate $\sim \tau_0^{-1}$. For infinite τ_1, both (5.2) and (5.4) give the overall decay time

$$\int_0^\infty G(t)\,dt \sim \tau_0,$$

but (5.4) badly approximates (5.2) at large t. In this last respect, the model differs from the scalar diffusion dynamics because the direct-interaction approximation is a good approximation for all t when the diffusing velocity field is frozen in time (Kraichnan 1970*a*).

To model the non-conservative evolution equation (1.1) for the magnetic field, we take the somewhat more complicated system

$$dy_1/dt = ay_2, \quad dy_2/dt = by_1, \quad dy_3/dt = ay_4, \quad dy_4/dt = -by_3. \tag{5.6}$$

Here a and b are normal random variables with zero means, constant on intervals $2\tau_1$ and statistically independent on distinct intervals. We take

$$\langle a^2\rangle = \langle b^2\rangle = 1/\tau_0^2, \quad \tau_0^2\langle ab\rangle = \alpha. \tag{5.7}$$

Here α models the mean helicity, and the subsystems (1, 2) and (3, 4) model the oppositely signed helicity modes of the magnetic field. The model system does not, in general, conserve Σy_n^2. Again, only diagonal elements of the mean response matrix are non-zero, and they satisfy

$$G_{11}(t) = G_{22}(t), \quad G_{33}(t) = G_{44}(t). \tag{5.8}$$

As before, consider $\tau_1 = \infty$ first. The exact values of the G matrix are

$$G_{11}(t) = \langle\cos[(-ab)^{\frac{1}{2}}t]\rangle, \quad G_{33}(t) = \langle\cos[(ab)^{\frac{1}{2}}t]\rangle. \tag{5.9}$$

The averages in (5.9) are easily evaluated for the three special cases $\alpha = 1$, 0 and -1. For $\alpha = 1$, we have $b = a$ in every realization, so that

$$\left.\begin{aligned} G_{11}(t) &= \langle\cosh(at)\rangle = \exp(\tfrac{1}{2}t^2/\tau_0^2), \\ G_{33}(t) &= \langle\cos(at)\rangle = \exp(-\tfrac{1}{2}t^2/\tau_0^2). \end{aligned}\right\} \tag{5.10}$$

For $\alpha = -1$, $b = -a$, and the values of G_{11} and G_{33} are exchanged. The case $\alpha = 0$ models the magnetic-field dynamics for reflexion-invariant isotropic turbulence.

Here there appears to be no simple closed form for the solution, but series expansion of (5.9) and application of the moment-evaluation rules for normal distributions yields the absolutely convergent series solution

$$G_{11}(t) = G_{33}(t) = Q(t/\tau_0), \quad Q(x) = \sum_{n=0}^{\infty} [(2n)!/2^n n!]^2 x^{4n}/(4n)!. \qquad (5.11)$$

$Q(t/\tau_0)$ rises monotonically with t, goes like $1+\frac{1}{24}(t/\tau_0)^4$ for small t, and goes like $\exp(\frac{1}{4}t^2/\tau_0^2)$ for large t.

If $0 < |\alpha| < 1$, then a and b are imperfectly correlated, with the result that both G_{11} and G_{33} in (5.10) are averages over mixtures of realizations with real frequencies and realizations with imaginary frequencies. Hence both become infinite as $t \to \infty$. If $1-\alpha \ll 1$, then G_{11} grows much faster than G_{33} as $t \to \infty$, and vice versa if $1+\alpha \ll 1$.

The arguments relating finite τ_1 to infinite τ_1 are the same as before. Thus we have

$$G_{11}(t) = \exp(t\tau_1/\tau_0^2), \quad G_{33}(t) = \exp(-t\tau_1/\tau_0^2) \quad (\alpha = 1,\ t = 2n\tau_1), \qquad (5.12)$$

$$G_{11}(t) = G_{33}(t) = [Q(2\tau_1/\tau_0)]^n \quad (\alpha = 0,\ t = 2n\tau_1). \qquad (5.13)$$

The direct-interaction equations for $\tau_1 = \infty$ are

$$dG_{11}/dt = \langle ab \rangle\, G_{11} * G_{11}, \quad dG_{33}/dt = -\langle ab \rangle\, G_{33} * G_{33}, \qquad (5.14)$$

where
$$G * G \equiv \int_0^t G(t-s)\, G(s)\, ds.$$

For $\alpha = 1$, the solution for G_{33} is given by (5.4) and that for G_{11} is obtained from it by replacing t by it. Thus

$$G_{11}(t) = (t/\tau_0)^{-1} I_1(2t/\tau_0), \quad G_{33}(t) = (t/\tau_0)^{-1} J_1(2t/\tau_0), \qquad (5.15)$$

where I_1 is the modified Bessel function of order 1. G_{11} behaves like $t^{-\frac{3}{2}}\exp(2t/\tau_0)$ for large t. For $\alpha = -1$, the expressions for G_{11} and G_{33} are exchanged. For $\alpha = 0$, it is immediately obvious from (5.14) that

$$G_{11}(t) = G_{33}(t) = 1 \quad (\text{all } t). \qquad (5.16)$$

The solutions of the direct-interaction equations for finite τ_1 are related to the infinite-τ_1 solutions as in the previous analysis. Thus

$$G_{11}(t) = [G_{11}(2\tau_1)]^n, \quad G_{33}(t) = [G_{33}(2\tau_1)]^n \quad (t = 2n\tau_1), \qquad (5.17)$$

where the right-hand sides are evaluated from (5.15) or (5.16).

Comparison of the exact solutions for this four-mode problem with the direct-interaction approximation, as presented in the preceding equations, shows that the approximations for both $G_{11}(t)$ and $G_{33}(t)$ are satisfactorily faithful, especially for finite τ_1, provided attention is confined to the cases $\alpha = 1$ and $\alpha = -1$. But for the case $\alpha = 0$, or uncorrelated a and b, the direct-interaction approximation gives the absurd result that G_{11} and G_{33} have the constant value one for all time. The failure of the approximation here is basically due to the fact that only second-order moments of a and b enter the equations (5.14) while the lowest moment which contributes to (5.11) is $\langle a^2b^2 \rangle$, which gives the t^4 term in $Q(t/\tau_0)$.

The present system is certainly too simple to model well all the relevant properties of the magnetic evolution equation (1.1). Nevertheless, the analogy would appear valid. The $\mathbf{B}.\nabla\mathbf{u}$ term in (1.1) supplies an uncorrelated contribution to the coupling coefficients of the modes of the $\mathbf{B}$ field in the case $\alpha = 0$, with the result that the direct-interaction approximation for the magnetic diffusion is identical with that for scalar diffusion, where the $\mathbf{B}.\nabla\mathbf{u}$ term is absent. The differences between the two cases do not show up until the fourth order of perturbation theory, and at that order they involve perturbation terms not included in the direct-interaction approximation. This has been pointed out previously by Roberts (1975). The model problem gives support to the double-ensemble approach to the case $\alpha = 0$ presented in §4. Thus a qualitatively satisfactory approximation to $G_{11}(t)$ and $G_{33}(t)$ for $\alpha = 0$ is given by the averages of the $G_{11}(t)$ and $G_{33}(t)$ values given by the direct-interaction approximations to the cases $\alpha = 1$ and $\alpha = -1$.

6. Discussion and conclusions

We wish now to summarize what is demonstrated and what is suggested in the analysis we have presented. First, in agreement with Roberts & Soward (1975) and in conflict with Moffatt (1974), we found in §3 that there is indeed an α-effect dynamo mechanism in helical isotropic turbulence with zero Ohmic diffusivity and with neglect of the Lorentz-force reaction on the turbulence. This α effect is predicted by the quasi-linear approximation and we find the latter to be asymptotically exact in the limit $\tau_1/\tau_0 \to 0$, where τ_1 and τ_0 are the correlation and eddy-circulation times of the turbulence respectively. An equivalent conclusion was reached by Steenbeck & Krause (1969).

In §4, we started by noting that the quasi-linear approximation, and the direct-interaction approximation as well, gave identical diffusion for magnetic and passive-scalar fields in reflexion-invariant isotropic turbulence. Further analysis showed that this result is internally inconsistent. A double-averaging procedure was carried out such that an overall reflexion-invariant ensemble was formed from realizations with locally coherent helicity fluctuations. It was found that the helicity fluctuations made an anomalous, negative contribution to the effective diffusivity acting on the mean magnetic field. If the helicity fluctuations are strong and if they persist over distances and times large compared with the turbulence correlation scales (but still as small as desired compared with the characteristic scales of the mean field), then it was found that the magnetic diffusivity is negative, so that the mean magnetic field grows unstably. At the same time there is no anomalous behaviour of the passive-scalar diffusivity. We may term this phenomenon the α^2 effect, since the anomalous diffusivity involves the correlation function of the locally smoothed α fluctuations.

The α^2 effect depends essentially on the helicity correlation, which is a fourth-order moment of the velocity field, while the ordinary turbulent diffusivity depends essentially on lower-order moments. A consequence is that the magnitude of the anomalous contribution to the diffusivity and, consequently, the size of the difference between magnetic and scalar diffusivities depend sensitively on the statistics of the turbulence. In §4 we attempted a crude estimate for normally distributed, isotropic, reflexionally invariant turbulence. This suggested that for such turbulence the α^2 effect is not large and that, in particular, the magnetic diffusivity is positive.

The existence of the α^2 effect shows that, in general, Parker's (1971) conclusion that the magnetic and scalar diffusivities at long times are equal in reflexionally invariant isotropic turbulence cannot be correct. In this connexion, we wish to present here a simple argument which shows that the asymptotic, long-time diffusivities are equal, in general, only if the diffusivities are equal for short times (of the order of the turbulence correlation time) also. We consider, as in the body of the text, velocity fields which are statistically independent on intervals of duration $2\tau_1$. Then the scalar-field Green's function for wavenumber k, which is diagonal in k because of homogeneity, must satisfy

$$g(k, 2n\tau_1) = [g(k, 2\tau_1)]^n \tag{6.1}$$

and the defining scalar of the magnetic-field Green's function must satisfy a similar relation. Hence the two Green's functions can be equal for $t = 2n\tau_1$ only if they are equal for $t = 2\tau_1$.

Because the helicity correlation function is of fourth order in the velocity field, the α^2 effect first shows up at the fourth order of perturbation theory or at the fourth order of a Taylor series expansion of the mean Green's tensor about $t = 0$. The coefficients in the direct-interaction equations are determined by second-order perturbation theory and this is why helicity-fluctuation effects do not show up in that approximation. The nature of the difficulty is modelled by the exactly soluble systems with several degrees of freedom examined in §5. It remains possible that the direct-interaction approximation gives reasonable accuracy when applied to magnetic diffusion in normal turbulence, where we estimated that the α^2 effect is not large.† If so, it might remain reasonably accurate when applied to the non-normal distributions of §4, provided the α fluctuations are on scales large compared with the turbulence correlation scale and provided the application is made in two stages. Thus the first stage would be to obtain (4.2), hopefully with reasonably accurate coefficients, and the second stage would be to take a superensemble, as explained in §4, and treat (4.2) by the direct-interaction approximation. The advantage over the quasi-linear approximation is that the restriction $\tau_1 \ll \tau_0$ and the similar restriction in (4.6) limiting the magnitude of the helicity correlation time τ_2 can be lifted.

Our treatment of the α^2 effect was carried out in the Eulerian formulation, where it appears to be simpler. However the results are confirmed by the analysis in the appendix, which uses Lagrangian expressions for α and for diffusivity which were derived by Moffatt (1974).

The sensitivity of the α^2 effect to the velocity-field statistics makes it desirable to develop accurate numerical methods of determination. In the case of isotropic turbulence whose statistics have an analytically expressible form, one possible procedure is to compute the magnetic Green's function scalar $g(k, t)$ as a power

† Straightforward Taylor expansion about $t = 0$ shows that the scalar and magnetic diffusivities in reflexion-invariant normal turbulence are in fact identical up to order t^4. Differences appear only at higher orders.

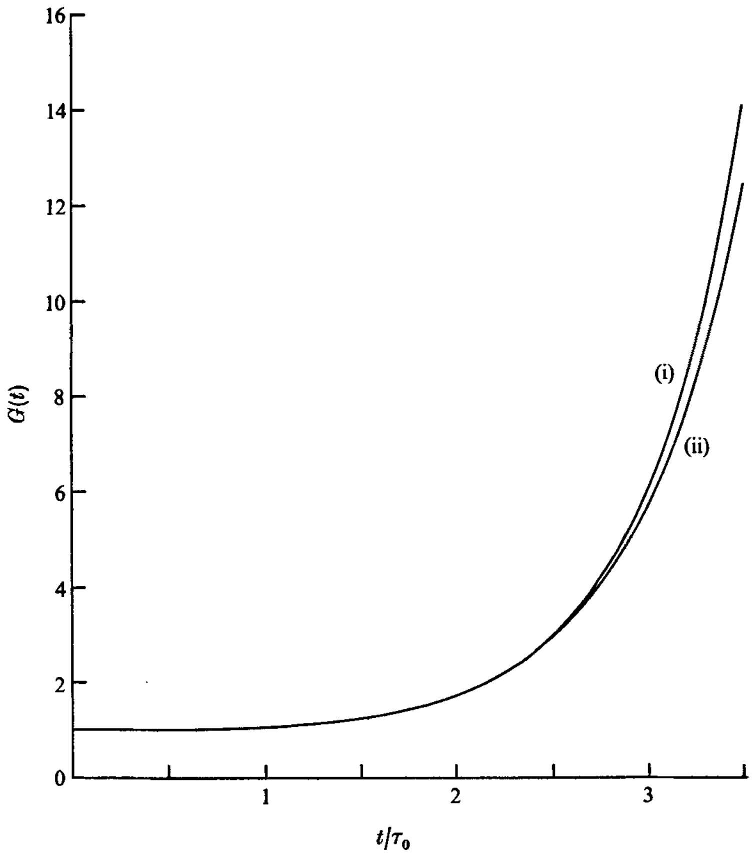

FIGURE 1. Model response function $G(t) = G_{11}(t) = G_{33}(t)$ as given (i) by the exact solution (5.11) and (ii) by the vertex approximation (6.2).

series in t. Although this series is divergent, convergents to $g(k, t)$ can be constructed from it. A similar procedure for finding the passive-scalar diffusivity in normal isotropic turbulence has proved successful (Kraichnan 1970*b*).

A more generally applicable approach, which holds promise for flows of astrophysical interest, is to compute the diffusion coefficients directly by following numerically the displacement and strain of an ensemble of fluid elements. We are currently carrying out such a programme for isotropic turbulence, using the diffusivity formulae derived in the appendix together with an adaptation of a computer program previously used to compute scalar diffusivity (Kraichnan 1970*a*). The preliminary results confirm the existence of the α^2 effect: the magnetic diffusivity is negative in reflexionally invariant isotropic turbulence with long-range, persistent helicity fluctuations. The results also show a negligible difference between scalar and magnetic diffusivities in normal, reflexionally invariant isotropic turbulence.

Finally, it is of interest to compute the magnetic diffusivity for isotropic turbulence by a higher-order closure approximation than the direct-interaction approximation. A higher closure involving vertex renormalization of the diagram expansion for the diffusion problem was described some years ago (Kraichnan 1961). This closure is exact through up to fourth-order terms in perturbation theory, and it was found to give extraordinarily accurate results for the random oscillator (5.1) (*loc. cit.*, figure 13). However, some global consistency properties of the direct-interaction approximation have not been proved for the higher closure.

The great accuracy of the vertex closure extends to the model system (5.6). In common with the exact solutions, the closure predicts that both $G_{11}(t)$ and $G_{33}(t)$ diverge as $t \to \infty$ unless $|\alpha| = 1$. For the crucial case $\alpha = 0$, the closure yields for $G = G_{11} = G_{33}$ the integral equations

$$\begin{aligned} dG/dt &= \tau_0^{-2} G * H, \quad G(0) = 1, \\ H &= \tau_0^{-2}(J*J*J + H*H*J), \\ J &= G + \tau_0^{-2}(J*J*H + H*H*H). \end{aligned} \tag{6.2}$$

Here H and J are vertex functions. We reserve the derivation of (6.2) for a later paper. The solution $G(t)$ is compared with the exact solution (5.11) in figure 1. It is in error by approximately 0·4 % at $t = 2\tau_0$ and by approximately 5 % at $t = 3\tau_0$. This indicates a high accuracy at all t if the closure is applied to the piecewise-constant case for $\tau_1 \leqslant \tau_0$.

It appears to be computationally feasible to carry out the vertex closure for scalar and magnetic diffusion by a normally distributed, isotropic, frozen (or piecewise constant) velocity field. An important outstanding question is how to include consistently the fourth-order cumulants of the velocity field which describe coherent helicity fluctuations in the non-normal case.

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Appendix. Lagrangian treatment of superensemble

The negative contribution to magnetic diffusivity from helicity fluctuations found by the Eulerian analysis of §4 is supported by the Lagrangian formulae for α and η obtained by Moffatt (1974). We shall start by rederiving Moffatt's results in a somewhat more compact form. Let the initial magnetic field $B_i(\mathbf{a}, 0)$ be expanded in a Taylor series about $\mathbf{x} = \mathbf{a} + \boldsymbol{\xi}$. Thus

$$B_i(\mathbf{a}, 0) = B_i(\mathbf{x}, 0) - \xi_n \, \partial B_i(\mathbf{x}, 0)/\partial x_n + \tfrac{1}{2}\xi_n \xi_m \, \partial^2 B_i(\mathbf{x}, 0)/\partial x_n \, \partial x_m - \dots . \tag{A 1}$$

Now take $\mathbf{x} = \mathbf{X}(\mathbf{a}, t)$, the position at time t of the fluid element which started at $\mathbf{a}$, and average over the ensemble. If $\mathbf{B}(\mathbf{a}, 0)$ is statistically independent of the velocity field, (A 1) used in (1.2) gives

$$\begin{aligned} \langle B_i(\mathbf{x}, t)\rangle = \langle \partial X_i/\partial a_j\rangle \langle B_j(\mathbf{x}, 0)\rangle - \gamma_{inj}\langle \partial B_j(\mathbf{x}, 0)/\partial x_n\rangle \\ + \zeta_{inmj}\langle \partial^2 B_j(\mathbf{x}, 0)/\partial x_n \, \partial x_m\rangle - \dots , \end{aligned} \tag{A 2}$$

where $$\gamma_{inj}(t) = \langle \xi_n(t) \, \partial X_i(t)/\partial a_j\rangle, \quad \zeta_{inmj}(t) = \tfrac{1}{2}\langle \xi_n(t) \, \xi_m(t) \, \partial X_i(t)/\partial a_j\rangle. \tag{A 3}$$

Homogeneity, isotropy, and the symmetry in n and m imply that the coefficients in (A 2) have the forms

$$\left.\begin{aligned} \partial X_i(t)/\partial a_j = \delta_{ij}, \quad \gamma_{inj}(t) = \epsilon_{inj}\gamma(t), \\ \zeta_{inmj}(t) = \delta_{ij}\delta_{nm}\zeta(t) + (\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn})\zeta'(t). \end{aligned}\right\} \tag{A 4}$$

By forming contractions of (A 4), we find

$$\gamma(t) = \tfrac{1}{6}\epsilon_{inj}\gamma_{inj}(t), \quad \zeta(t) = \tfrac{1}{15}[2\zeta_{inni}(t) - \zeta_{iinn}(t)]. \tag{A 5}$$

The terms involving $\zeta'(t)$ cannot contribute in (A 2) because $\nabla.\mathbf{B} = 0$. The averages in (A 2) and (A 3) are taken at fixed $\mathbf{x}$. If the forms (A 4) are used in (A 2), the result is

$$\langle \mathbf{B}(\mathbf{x},t)\rangle = \langle \mathbf{B}(\mathbf{x},0)\rangle - \gamma(t)\,\nabla\times\langle \mathbf{B}(\mathbf{x},0)\rangle + \zeta(t)\,\nabla^2\langle \mathbf{B}(\mathbf{x},0)\rangle - \dots. \tag{A 6}$$

It follows from isotropy that $\gamma(t)$ and $\zeta(t)$ can be written as

$$\gamma(t) = \langle \xi_2\,\partial\xi_1/\partial a_3\rangle, \quad \zeta(t) = \tfrac{1}{2}\langle \xi_2^2(1 + \partial\xi_1/\partial a_1)\rangle. \tag{A 7}$$

A similar treatment of the scalar field yields

$$\langle \phi(\mathbf{x},t)\rangle = \langle \phi(\mathbf{x},0)\rangle + \zeta_s(t)\,\nabla^2\langle \phi(\mathbf{x},0)\rangle - \dots, \tag{A 8}$$

where

$$\zeta_s(t) = \tfrac{1}{6}\langle \xi_i\,\xi_i\rangle = \tfrac{1}{2}\langle \xi_2^2\rangle. \tag{A 9}$$

Now differentiate (A 6) with respect to t. Following Moffatt (1974), we can express each $\langle \mathbf{B}(\mathbf{x},0)\rangle$ on the right-hand side of the resulting differential equation as an infinite series in $\langle \mathbf{B}(\mathbf{x},t)\rangle$ and its spatial derivatives by an iteration-reversion of (A 6). The result is

$$\partial\langle \mathbf{B}(\mathbf{x},t)\rangle/\partial t = -\alpha(t)\,\nabla\times\langle \mathbf{B}(\mathbf{x},t)\rangle + \eta(t)\,\nabla^2\langle \mathbf{B}(\mathbf{x},t)\rangle - \dots, \tag{A 10}$$

where

$$\alpha(t) = d\gamma(t)/dt, \quad \eta(t) = d\zeta(t)/dt + \tfrac{1}{2}d[\gamma(t)]^2/dt. \tag{A 11}$$

By using isotropy properties and the homogeneity relation

$$\partial\langle \xi_n\,\xi_m \dots v_s \dots\rangle/\partial a_j = 0$$

it is easy to show that the coefficients (A 11) for the α effect and diffusivity are the same as those obtained by Moffatt. [Moffatt's $\alpha(t)$ is defined as the negative of ours; also, there are errors of sign in equations (4.4) and (4.6) of his paper.]

If the spatial variation of $\langle \mathbf{B}(\mathbf{x},t)\rangle$ is slow enough, the higher terms in the series (A 10) should be negligible compared with the terms shown explicitly. Assume that this is so and consider two isotropic homogeneous ensembles which are identical except that they give equal and opposite $\alpha(t)$. Homogeneity requires that $\alpha(t)$ and $\eta(t)$ be independent of position. At any t, the dispersion of fluid elements is finite, however, and the strictly homogeneous distribution will be locally indistinguishable, in its effects on the magnetic field, from distributions in which $\alpha(t)$ varies on a spatial scale large compared with the dispersion. By symmetry $\eta(t)$ is independent of the sign of $\alpha(t)$. Let $\eta_\alpha(t)$ denote the value of $\eta(t)$ in either of the two ensembles. Let $\eta_0(t)$ denote the value of $\eta(t)$ in the superensemble obtained by combining the positive-α and negative-α ensembles with equal weight. The analysis leading to (A 10) goes through for the superensemble in precisely the same way as for the individual ensembles. The values of $\gamma(t)$ and $\zeta(t)$ are simply the averages of the values for the individual ensembles. Hence $\gamma(t) = 0$ in the superensemble while $\zeta(t)$ has the same value as it has in the two individual ensembles. It then follows from (A 11) that

$$\eta_0(t) = \eta_\alpha(t) - \tfrac{1}{2}d[\gamma(t)]^2_\alpha/dt = \eta_\alpha(t) - \alpha(t)\int_0^t \alpha(s)\,ds. \tag{A 12}$$

Now if we retrace the derivation of (4.8) without taking $t \gg \tau_2$, the result is

$$\partial\langle\mathbf{B}\rangle/\partial t = \left[\eta - \int_0^t \langle\alpha(t)\,\alpha(s)\rangle\right]\nabla^2\langle\mathbf{B}\rangle. \tag{A 13}$$

Thus, identifying $\eta_0(t)$ with the coefficient of $\nabla^2\langle\mathbf{B}\rangle$ in (A 13) and identifying $\eta_\alpha(t)$ with the Eulerian subensemble diffusivity η in (4.2), we see that both the Eulerian and the Lagrangian formulation yield the same relationship between subensemble and superensemble diffusivities. The additional fact given by the Eulerian analysis is that for $\tau_1 \ll \tau_0$ the diffusivity η for the subensemble is independent of the helicity parameter μ. It is less simple to derive this fact from the Lagrangian analysis, unless the latter is applied independently to each interval $2\tau_1$. When $t \to \infty$ at constant α, the $d[\gamma(t)]^2/dt$ term in (A 11) becomes infinite as noted by Moffatt (1974). The Eulerian analysis implies that $d\zeta(t)/dt$ becomes negatively infinite, so that $\eta(t)$ remains finite.

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